Cost-sensitive computational adequacy of higher-order recursion in synthetic domain theory

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Yue Niu Jon Sterling Robert Harper

Carnegie Mellon University ✓ yuen@cs.cmu.edu Cost-sensitive synthetic domain theory 00000000000 CONCLUSION 0000

References

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Introduction

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The story begins with a type theory **calf** developed to unify cost-sensitive and functional verification [Niu+22].

- Functional: IO-behavior of programs, data structure invariants
- Cost-sensitive: computational cost or resource usage (time, space, etc.)

Functional properties are about if a program is correct, cost-sensitive properties are about how much resource a program uses.

Introduction

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calf supports a *denotational* style of cost analysis — connection to operational semantics via a cost-sensitive computational adequacy property à la Plotkin [Plo77].

Prior work: cost-sensitive adequacy for first-order recursion [NH23].

This talk: cost-sensitive adequacy for **PCF** (*higher-order*) recursion).

Outline

Introduction to **calf**:

- Cost-sensitive and functional reasoning in **calf**
- Cost-sensitive adequacy property

Integrating higher-order recursion in **calf**:

- Introduction to *synthetic domain theory* (SDT)
- Cost-sensitive SDT
- Cost-sensitive adequacy in SDT

Cost as an abstract effect

In **calf**, cost is an *abstract* effect F(A) supporting an operation step : $\mathbb{C} \to F(1)$. Think of step^{*c*} as taking *c* abstract steps:

```
insertSort : list \rightarrow F(list)
insertSort(l) = ... step<sup>c</sup>; e...
```

Under the hood define $F(A) = \mathbb{C} \times A$ and $step^c = (c, \star)$. Can reason about step's equationally:

```
step<sup>c_1</sup>; step<sup>c_2</sup> = step<sup>c_1+c_2</sup>
step<sup>o</sup>; e = e
```

Functional reasoning in calf

How to reason about the purely *functional* properties of cost-sensitive programs?

 $isSorted(insertSort(l)) \iff isSorted(mergeSort(l))$

Should be automatic because both are sorting algorithms. But *not* because *insertSort* \neq *mergeSort* due to presence of cost structure!

Cost as a phase

The functional semantics of (total) programs is naturally modeled in **Set**.

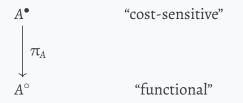
Set is too "flat": the cost effect $\mathbb{C} \times -:$ Set \to Set does not distinguish data from cost structure.

calf: cost as a new dimension or phase.

Cost structure as families

calf = the internal type theory of the category of *families* **Set** \rightarrow

A type in Set^{\rightarrow} is a cost-sensitive set equipped with a restriction action to the purely functional component:



Think Kripke/possible worlds semantics over $\mathbb{I} = \{ \circ \to \bullet \}$.

Functional vs. cost-sensitive phase

Presheaves over $\{\circ \rightarrow \bullet\}$ exhibits a **phase distinction**:

- World at \circ = **functional phase**
- World at = **cost-sensitive phase**
- In cost-sensitive phase, *insertSort* \neq *mergeSort*.
- In functional phase, *insertSort* = *mergeSort*.

Presheaf restriction $\bullet \rightarrow \circ$ trivializes/redacts cost structure!

Modal types

A modal type is either purely functional or purely cost-sensitive.

Definition

A type is *purely functional* or *function-modal* when it is in the image of the constant presheaves functor $\mathbf{Set} \to \mathbf{Set}^{\to}$.

Definition

A type is *purely cost-sensitive* or *cost-modal* when it is given by a terminal map $A \rightarrow 1$.

Cost effect with cost-modal types

Define F(A) by using a *cost-modal* monoid object \mathbb{C} :

$$\mathsf{F}(\begin{array}{c} A^{\bullet} \\ \downarrow \\ A^{\circ} \end{array}) = \begin{array}{c} \mathbb{N} \\ \downarrow \\ 1 \end{array} \times \begin{array}{c} A^{\bullet} \\ \downarrow \\ A^{\circ} \end{array} = \begin{array}{c} \mathbb{N} \times A^{\bullet} \\ \downarrow \\ A^{\circ} \end{array}$$

Restriction deletes cost structure.

Internalization

Modal types can be phrased in the internal language of $\mathbf{Set}^{\rightarrow}.$

Let \P : Ω be the *intermediate* proposition in **Set**^{\rightarrow}:

Assuming \P = restricting to the functional phase of **calf**.

Internal characterization of modal types

Proposition

A type A is function-modal when $(\P \to A) \cong A$.

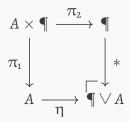
Proposition

A type A is cost-modal when $(\P \to A) \cong \mathbf{1}$.

In other words, a function-modal type "thinks" the functional phase holds and a cost-modal type "thinks" the functional phase is false.

Constructing modal types

Given A, $\P \to A$ is function-modal. Dually, construct a cost-modal type $\P \lor A$ as follows:



The *cost modality* $\P \lor -$ quotients the type *A* to a unique point * in the functional phase.

Functional and cost reasoning, internally

Semantically, $F(A) = (\P \lor \mathbb{C}) \times A$.

Thus *insertSort* \neq *mergeSort* since the cost monoid $\P \lor \mathbb{C}$ is nontrivial.

But, $\P \to ((\P \lor \mathbb{C}) \cong 1)$, so insertSort = mergeSort in the functional phase!

calf vs. programming languages

Cost analysis in **calf** is equational or denotational $(step^{c_1}; step^{c_2} = step^{c_1+c_2})$.

Problems:

- How to relate cost analysis in **calf** to PLs with *operational* cost semantics?
- How to reconcile general recursive functions in PLs with total functions in **calf**?

calf vs. programming languages

Solution:

- Enrich **calf** with *partiality* via *synthetic domain theory*.
- Relate PLs and **calf** by an *internal, cost-sensitive* computational adequacy property.

Upshot:

- General recursive programming in **calf**
- Cost-sensitive generalization of Plotkin's classic adequacy property.

Cost-sensitive computational adequacy

Example: take STLC equipped with the cost effect F(A). Internal to **calf**, we have a language $\mathcal{L} = (Ty : \mathcal{U}, Tm : Ty \to \mathcal{U})$.

Internal denotational cost semantics of \mathcal{L} :

- $\bullet \ \llbracket \rrbracket_{Ty} : Ty \to \mathcal{U}$
- $(\llbracket \rrbracket_{\mathsf{Tm}})_A : \mathsf{Tm}(A) \to \llbracket A \rrbracket_{\mathsf{Ty}}$

As before $\llbracket F(A) \rrbracket = \mathbb{C} \times \llbracket A \rrbracket$.

Internal operational cost semantics of \mathcal{L} :

• $\Downarrow_A \subseteq \mathsf{Tm}(A) \times \mathbb{C} \times \mathsf{Tm}(A)$

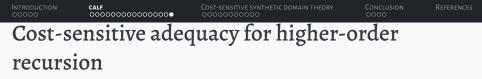
Cost-sensitive computational adequacy

Definition

A language satisfies **cost-sensitive computational adequacy** when for all $e : F(2), [\![e]\!] =_{\mathbb{C} \times [\![A]\!]} (c, [\![v]\!])$ if and only if $e \Downarrow^c v$.

Classic Plotkin adequacy: $[\![-]\!]$ carves out functions that are definable operationally.

Cost-sensitive adequacy: [-] carves out **calf** functions that are definable operationally *in a cost-reflecting way*.



Prior work: $\mathcal{L} = \text{Algol-like}$ languages with while loops [NH23].

This work: $\mathcal{L} = \mathbf{PCF}$.

Recursion in type theory

To define the denotational cost semantics of **PCF** in **calf**, we need a notion of *partial* functions in type theory.

Attempt: model **calf** in presheaves valued in ω cpo's: ω CPO $^{\rightarrow}$.

Unfortunately not a model of dependent type theory.

Synthetic domain theory

Integrate higher-order recursion into type theory by means of *synthetic domain theory* (SDT):

- Intuitionistic type theory
- Class of predomains
- All definable predomain maps automatically continuous

Concretely: a topos $\boldsymbol{\mathcal{E}}$ equipped with a full subcategory Predom.

Axioms of SDT

To start, we need an object called the *dominance* that serves as the classifier of the *support* of partial maps.

Definition

A dominance subobject $\Sigma \hookrightarrow \Omega$ that is closed under $\top:\Omega$ and dependent sums.

Frequently Σ is also required to be closed under $\bot:\Omega.$

Lifting structure

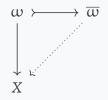
The dominance Σ induces a *lifting structure* $L(A) = \Sigma_{\phi:\Sigma} \cdot \phi \to A$: partial maps $A \xleftarrow{\Sigma} D \to B$ as total maps $A \to L(B)$.

Lifting induces an incidence relation $\omega \hookrightarrow \overline{\omega}$ including the initial lift algebra ω into the final lift coalgebra $\overline{\omega}$.

Think of $\omega \hookrightarrow \overline{\omega}$ as a *figure shape* that we use to state the completeness properties of predomains.

Predomains in SDT

A *predomain* has the unique extension property along $\omega \hookrightarrow \overline{\omega}$:



Synthetic counterpart to ω cpos, which extend along the figure shape $\{0 \leq 1 \leq \ldots\} \hookrightarrow \{0 \leq 1 \leq \cdots \leq \infty\}$.

Model of SDT

A model of SDT is given by a topos \mathcal{E} equipped with a predomain dominance Σ .

Every such model induces a full subcategory of predomains that is a *reflective exponential ideal*:

- Closed under limits and exponentials: types of **PCF**
- All colimits exist: used to define the cost-modal type $\P \vee \mathbb{C}$
- Every endomap of domains (predomains with lift algebras) has a fixed-point: fix operator

To interpret **PCF** with the cost effect F(A), need a proposition ϕ for the *functional phase*:

Definition

A model of SDT with a phase distinction is a model of SDT $(\mathcal{E}, \Sigma, \varphi)$ where φ is a Σ -proposition.

Semantically: $\llbracket F(A) \rrbracket = L(\mathbb{C} \times \llbracket A \rrbracket)$ with \mathbb{C} cost-modal.

Need $\phi : \Sigma$ to ensure $\phi \lor A$ is a predomain when A is one.

Operational semantics of **PCF**

Our proof of computational adequacy relies on the fact that $e \Downarrow^c v$ is a Σ -proposition.

Define the operational semantics as a partial function eval : $Tm(F(A)) \rightarrow Tm(F(A)) \rightarrow L(\mathbb{C})$:

$$eval(e, v) = \begin{cases} c \boxplus eval(e', v) & out(e) = inr \cdot (c, e') \\ (e = v, \lambda u.o) & out(e) = inl \cdot \star \end{cases}$$

In the above, we write out : $Tm(A) \rightarrow 1 + (\mathbb{C} \times Tm(A))$ for the one step transition relation, and $- \boxplus -$ for the cost algebra map.

Logical relation for computational adequacy

Define a family of relations $\triangleleft_A \subseteq \llbracket A \rrbracket \times \mathsf{Tm}(A)$ between the syntax and semantics of **PCF**.

A technical point is the definition of $\triangleleft_{F(A)}$:

$$e (R \Rightarrow S) e' = \forall [a R a'] (e a) S (e' a')$$

$$e \triangleleft_{FA} e' = \forall [f (\triangleleft_A \Rightarrow \leqslant) f'] e; f \leqslant e'; f'$$

In the above we write $e \leq e'$ for the specialization order or definedness order on $F(1) \cong L(\mathbb{C})$.

Ensures that $(-\triangleleft_{F(A)} e') \subseteq [\![F(A)]\!]$ is always a sub-predomain or *admissible*.

INTRODUCTION CALF CONCLUSION COST-SENSITIVE SYNTHETIC DOMAIN THEORY CONCLUSION REFER Fundamental lemma and computational adequacy

We may prove the fundamental lemma of the logical relation:

Theorem Given $\Gamma \vdash e : A$, we have $\Gamma \vdash \llbracket e \rrbracket \lhd_A e$.

Cost-sensitive computational adequacy follows directly from the fundamental lemma:

Theorem

Given e : F(1), we have that $\llbracket e \rrbracket = eval(e, \star)$.

Model of cost-sensitive SDT

To incorporate cost structure as a phase distinction, define a model of SDT fibred over $\mathbf{Set}^{\rightarrow}$.

Isolate a (small) category \mathbb{C} of *internal dcpos* in **Set** \rightarrow .

- Presheaves on C is *almost* a model of SDT.
- Restrict to sheaves on $\mathbb C$ for the extensive coverage: preserves \emptyset and +.

Theorem

The category of (internal) sheaves on C furnishes a model of SDT such that the functional phase proposition \P : C is preserved by the Yoneda embedding.

Related work

- Computational adequacy in SDT [Sim99; Sim04]
- Relative sheaf models of SDT [SH22]
- Rooted in the type-theoretic framework **calf**
- Extended the results of Niu and Harper [NH23] to **PCF**
- Denotational cost semantics based on prior work on effectful **PCF** [Kav+19]

Conclusion

- Integrated higher-order recursion into **calf** type theory
- Internal cost-sensitive computational adequacy theorem for **PCF**
- Connecting denotational and operational reasoning for cost analysis in type theory
- Relative sheaf model of the function-cost phase distinction

Future work

- Recursive types [Sim04]
- Relating internal and *external* cost-sensitive adequacy

Thanks for listening!

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